4.3.4 Illustrative Example of the Zionts and Wallenius Method - Continuation

Second iteration

Step 2. The optimal simplex table of (P_{λ^2}) regarding $\mathbf{x}^{(2)}$ is:

	$\mathbf{c}$	1.061	1.646	1.677	2.293	0	0	0	
$(\mathbf{c}_B)^T$	$\mathbf{x}_B$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1.677	x_3	0	-1	1	0	0.5	-0.25	-0.25	2.5
1.061	x_1	1	1	0	0	0.1	0.35	-0.25	14.5
2.293	x_4	0	1	0	1	-0.4	0.1	0.5	7
$z_j^1 - c_j^1$		0	1	0	0	0.9	0.65	-0.75	55.5
$z_j^2 - c_j^2$		0	4	0	0	-0.5	0.25	1.25	47.5
$z_j^3 - c_j^3$		0	-5	0	0	-0.4	-0.4	1	2
$z_j^\lambda - c_j^\lambda$		0	0.031	0	0	0.028	0.182	0.461	

The corresponding reduced cost matrix regarding the nonbasic variables x_2, x_5, x_6 and x_7 is:

$$\mathbf{W} = \begin{bmatrix} 1 & 0.9 & 0.65 & -0.75 \\ 4 & -0.5 & 0.25 & 1.25 \\ -5 & -0.4 & -0.4 & 1 \end{bmatrix}$$

After applying the Zionts-Wallenius routine to the matrix **W**, it is concluded that, among x_2, x_5, x_6 and x_7, only x_2, x_5 and x_7 are efficient (corresponding to the 1st, 2nd and 4th columns of **W**).

In order to determine whether x_2 should belong to the set *A* or the set *B*, it is just necessary to verify if the system composed by the inequalities of $\Lambda^{(2)}$, with $\begin{bmatrix}\lambda_1 \; \lambda_2 \; \lambda_3\end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} \le \varepsilon$ (with ε very small, for example, 10^{-5}), has, or not, a solution. If it has a solution then x_2 belongs to *A*, otherwise x_2 belongs to *B*. This test can be made using the first phase of the simplex method.

The procedure is repeated for x_5 and x_7.

In this case, $A = \{x_2, x_7\}$ and B=$\{x_5\}$, that is, for the current solution, if x_2 or x_7 become basic variables then the computation leads to the efficient vertices reachable with weights within $\Lambda^{(2)}$. (Note that the same does not happen with x_5.)

Set $I = A$.

Step 3 (for I = A). All the efficient solutions adjacent to $(\mathbf{x}^{(2)}, \mathbf{z}^{(2)})$ are generated, corresponding to the set I.

- x_2 becomes a basic variable and z^{adj1} = (48.5, 19.5, 37) is obtained. This solution is presented to the DM, who indicates his/her preference between $\mathbf{z}^{adj1}$ and the current solution $\mathbf{z}^{(2)}$.

  Suppose that the DM's answer is no, that is, he/she prefers $\mathbf{z}^{(2)}$.

- x_7 becomes a basic variable and $\mathbf{z}^{adj2}$ = (66, 30, -12) is obtained.

  Suppose that the DM does not know which one of the two solutions, $\mathbf{z}^{(2)}$ or $\mathbf{z}^{adj2}$, he/she prefers.

Step 4 (for I = A). Although no adjacent solution to $\mathbf{z}^{(2)}$ is preferred during step 3, all the efficient basic solutions obtained using the variables in the set I have already been evaluated (because all the adjacent solutions are sufficiently distinct from the current solution). Hence, the algorithm proceeds to step 5.

Step 5 (for I=A). In this step the DM has the opportunity to identify the variation trends of the objective functions associated with edges emanating from the current solution that are acceptable. This is done in step 3, although the vertices those edges lead to are not preferred to the current solution.

Regarding the set I, the DM is asked to evaluate the variation trends (*trade-offs*), that is, the variation of the objective functions by unit of the nonbasic variable that becomes basic:

- *Trade-off* (1, 4, -5), column of x_2 in **W**: meaning that for each unit of the nonbasic variable x_2 there is a decrease of 1 in z_1, a decrease of 4 in z_2, and an increase of 5 in z_3 (Fig. A.1).

  Suppose that the DM does not accept this variation trend.

- *Trade-off* (-0.75, 1.25, 1), column of x_7 in **W**: meaning that for each unit of the nonbasic variable x_7 there is an increase of 0.75 in z_1, a decrease of 1.25 in z_2, and a decrease of 1 in z_3 (Fig. A.2).

Fig. A.1 – Evaluation of the *trade-off* (1, 4, -5).

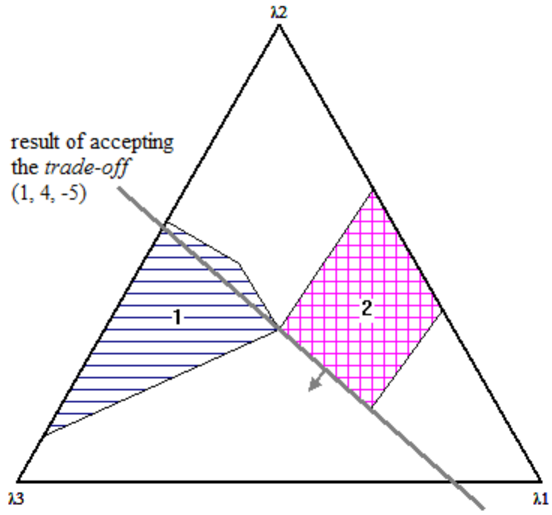
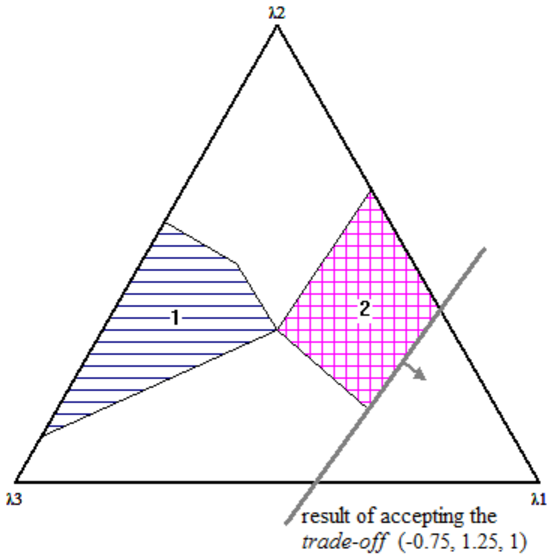


Fig. A.2 – Evaluation of the *trade-off* (-0.75, 1.25, 1).



Suppose that the DM <u>does not</u> also accept this variation trend.

Step 6. Set $I = B = \{ x_5 \}$, meaning that the other set of efficient basic variables (which lead to efficient solutions that cannot be reached using weights in the reduced weight space of the current iteration) should be analyzed.

Return to step 3.

Step 3 (for $I = B$). The efficient solution adjacent to $(\mathbf{x}^{(2)},\mathbf{z}^{(2)})$ is computed, corresponding to making x_5 a basic variable (the only variable of the set I): $\mathbf{z}^{adj3}$ = (51, 50, 4) is obtained. This solution is presented to the DM, and he/she indicates if he/she prefers $\mathbf{z}^{adj3}$ to the current solution $\mathbf{z}^{(2)}$.

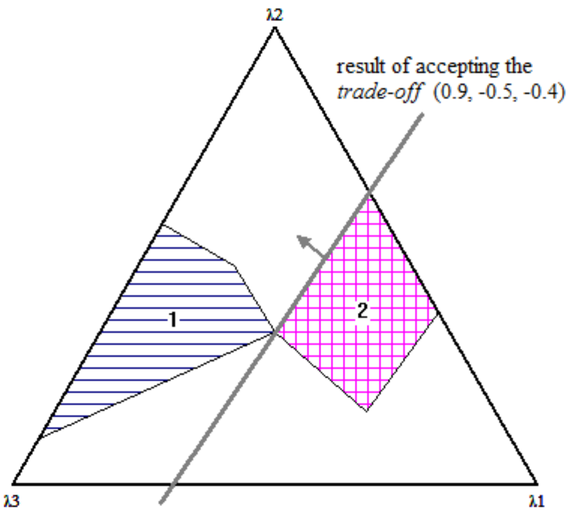
Suppose that the DM's answer is does not know, that is, he/she is not able to express a preference between $\mathbf{z}^{adj3}$ and $\mathbf{z}^{(2)}$.

Step 4 (for $I = B$). Regarding the set I, there is no adjacent efficient solution which has not been evaluated in step 3 (notice that a solution would not have been evaluated if it was not sufficiently distinct from $\mathbf{z}^{(2)}$). Hence, the algorithm goes to step 5.

Step 5 (for $I = B$). Regarding the set I, the DM is asked to evaluate the variation trend corresponding to the displacement along the edge emanating from the current efficient vertex leading to the solution that was neither accepted nor rejected (for I=B) in step 3, $\mathbf{z}^{adj3}$:

- *Trade-off* (0.9, -0.5, -0.4), column of x_5 in **W**: meaning that for each unit of the nonbasic variable x_5 there is a decrease of 0.9 in z_1, an increase of 0.5 in z_2 and an increase of 0.4 in z_3 (Fig. A.3)

Fig. A.3 – Evaluation of the *trade-off* (0.9, -0.5, -0.4).



Suppose that the DM accepts this variation trend.

At this stage of the interactive process, there are neither positive answers in pairwise comparisons in the two times it went through step 3 nor any variation trends of the objective functions have been accepted corresponding to the nonbasic variables belonging to A. Nevertheless, a variation trend corresponding to a nonbasic variable from set B has now been accepted. Hence, the algorithm moves to step 7.

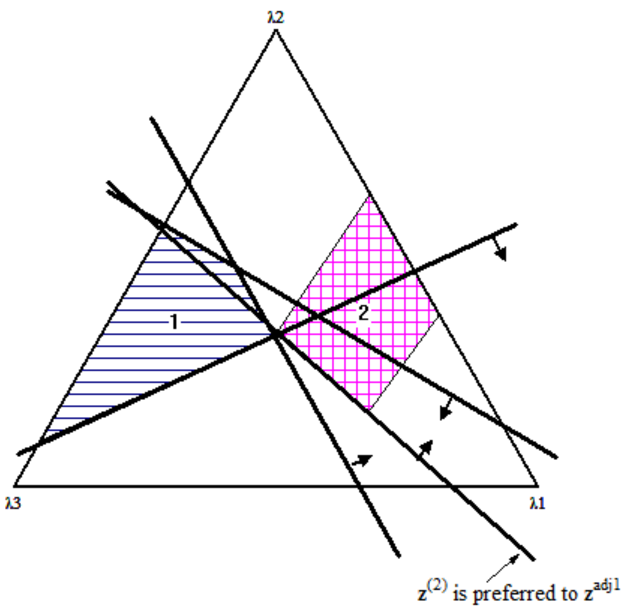
Step 7. Constraints resulting from the answers given by the DM in steps 3 and 5 (for $I = A$ and $I = B$) are added to the weight space, which will form the reduced weight space $\Lambda^{(3)}$.

- The answers given by the DM in step 3 ($I = A$) are the following:
  The first answer is <u>no</u>, that is, the DM does not prefer $\mathbf{z}^{adj1}$ to $\mathbf{z}^{(2)}$; this leads to the constraint:

$$\lambda(\mathbf{z}^{(2)} - \mathbf{z}^{adj1}) \geq \varepsilon \Leftrightarrow \begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}\left(\begin{bmatrix}55.5\\47.5\\2\end{bmatrix} - \begin{bmatrix}48.5\\19.5\\37\end{bmatrix}\right) \geq \varepsilon \quad \text{(see Fig. A.4)}$$

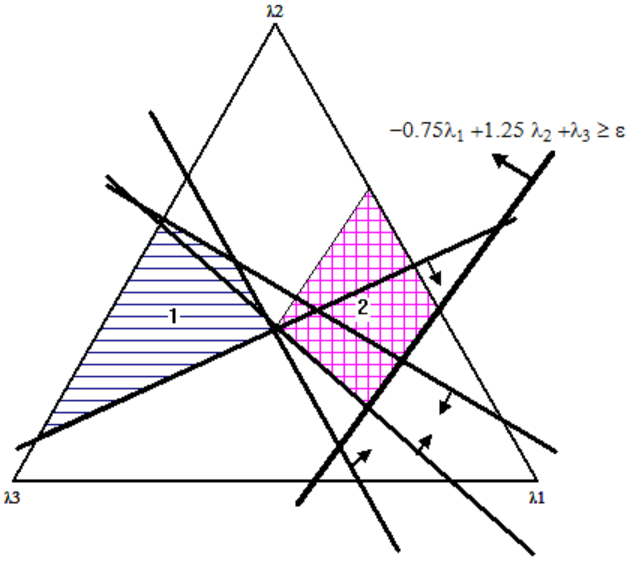
The second answer of the DM is <u>does not know</u> (that is, the DM is not able to express his/her preference between $\mathbf{z}^{adj2}$ and $\mathbf{z}^{(2)}$); therefore, no constraint is introduced on the weight space.

Fig. A.4 – Result of the comparison between the solutions $\mathbf{z}^{adj1}$ and $\mathbf{z}^{(2)}$.



- In the step ($I = A$), the DM gave the following answers:
  The DM <u>did not</u> accept the *trade-off* (1, 4, -5), thus resulting in the introduction of constraint $\lambda_1 + 4\lambda_2 - 5\lambda_3 \geq \varepsilon$, which coincides with the constraint imposed by the preference of $\mathbf{z}^{(2)}$ over $\mathbf{z}^{adj1}$ (Fig. A.4).
  The DM <u>did not</u> accept the *trade-off* (-0.75, 1.25, 1), thus resulting in the introduction of constraint $-0.75\lambda_1 + 1.25\lambda_2 + \lambda_3 \geq \varepsilon$ (figure A.5).

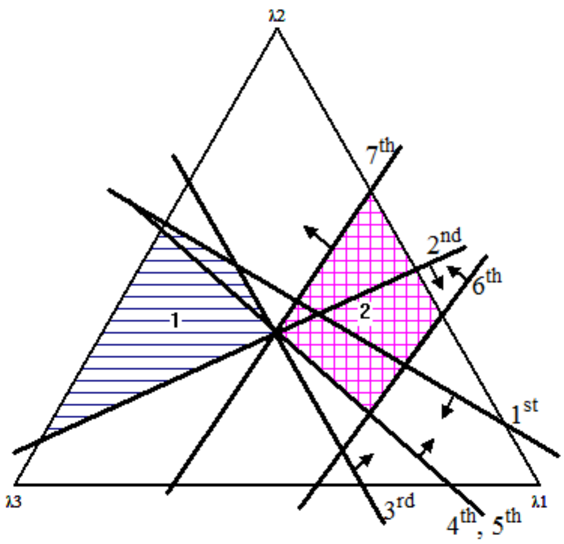
Fig. A.5 – Evaluation of the *trade-off* (-0.75, 1.25, 1).



- The answer given in step ($I = B$) was the following:
  The DM said that he/she <u>did not know</u>, that is, he/she was not able to express his/her preference between $\mathbf{z}^{adj3}$ and $\mathbf{z}^{(2)}$. Therefore, no new constraint is created on the weight space.
- The answer given in step 5 ($I = B$) was the following:
  The DM <u>accepted</u> the *trade-off* (0.9, -0.5, -0.4). This answer results in the introduction of constraint: $-0.9\lambda_1 + 0.5\lambda_2 + 0.4\lambda_3 \geq \varepsilon$ (Fig. A.6, 7th constraint).

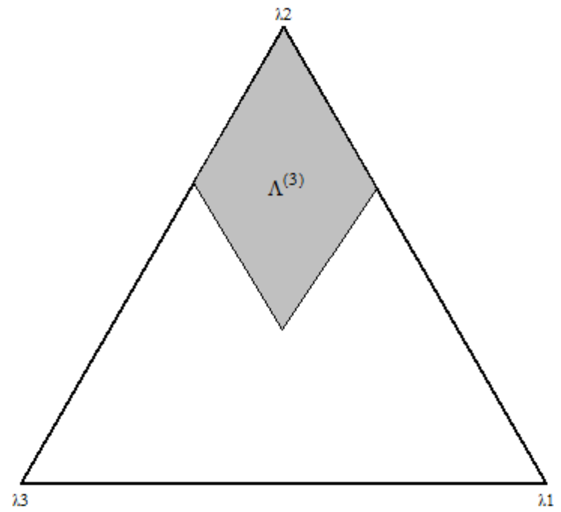
Fig. A.6 shows all constraints considered until now and the order by which they were introduced (the first three constraints correspond to the 1st iteration and the remaining ones to the 2nd iteration). The set of all constraints defines $\Lambda^{(3)}$, which is an empty set in this case.

Fig. A.6 – Constraints introduced on the weight space.



Step 8. It is not possible to determine $\boldsymbol{\lambda}^{(3)} \in \Lambda^{(3)}$ because $\Lambda^{(3)} = \varnothing$. Therefore, the oldest constraint is eliminated. Even so $\Lambda^{(3)} = \varnothing$. Then the 2nd oldest constraint is eliminated, and the result is $\Lambda^{(3)} \neq \varnothing$ (Fig. A.7).

Fig. A.7 – $\Lambda^{(3)}$ after the elimination of the two oldest constraints.



A point $\boldsymbol{\lambda}^{(3)} \in \Lambda^{(3)}$ is determined by solving the problem to maximize the least deviation to all the constraints defining $\Lambda^{(3)}$. The solution to this problem is $\boldsymbol{\lambda}^{(3)} = (0.2, 0.6, 0.2)$.

Step 9. Problem (P_{λ^3}) using $\boldsymbol{\lambda}^{(3)}$ is solved:

$$\begin{aligned} \max \quad & 0.2\,(3x_1 + x_2 + 2x_3 + x_4) \\ & +0.6\,(x_1 - x_2 + 2x_3 + 4x_4) \\ & +0.2\,(-x_1 + 5x_2 + x_3 + 2x_4) \\ \text{s. t.} \quad & 2x_1 + x_2 + 4x_3 + 3x_4 \le 60 \\ & 3x_1 + 4x_2 + x_3 + 2x_4 \le 60 \\ & x_1 + 2x_2 + 3x_3 + 4x_4 \le 50 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned} \qquad (P_{\lambda^3})$$

The solution obtained is $\mathbf{x}^{(b)}$ = (14, 0, 0, 9) with $\mathbf{z}^{(b)}$ = (51, 50, 4).

Step 10. The DM is asked to express his/her preference between $\mathbf{z}^{(b)}$ and the current solution, $\mathbf{z}^{(2)}$ = (55.5, 47.5, 2).

Note that it is not the first time that the DM compares these two solutions, since $\mathbf{z}^{(b)}$ was also the solution found in step 3 for $I = B$ ($\mathbf{z}^{adj3}$). At that point the DM was not able to express a preference between these solutions. Suppose that now the DM chooses $\mathbf{z}^{(2)}$. Note that, in this phase of the procedure, the DM should express his/her preference.

Therefore, the method stops and $\mathbf{x}^{(2)}$ = (14.5, 0, 2.5, 7), $\mathbf{z}^{(2)}$ = (55.5, 47.5, 2) is the final solution.